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Improved Version of Gram-Schmidt Method for Modal Analysis of Structures

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Abstract

This paper presents an innovative eigenvalue solution technique that leverages an enhanced Gram-Schmidt procedure to impr ove the efficiency and stability of QR-based methods. Among iterative techniques, such as vector, polynomial, and subspace i terations, transformation methods generally demonstrate superior performance. Specifically, the Jacobi and Householder QR methods are well-established in finite element analysis due to their high computational efficiency. Although the Householder QR method is preferred for larger systems owing to its initial reduction of the matrix to tridiagonal form, it possesses a critica l drawback: the reduction process increases the bandwidth of the unreduced matrix, thereby disrupting the efficient exploitati on of the original banded structure of matrix K. To overcome this limitation, we propose an enhanced version of the Gram-Sc hmidt procedure. Unlike the standard Modified Gram-Schmidt (MGS) algorithm, our approach incorporates a strategic projec tion mechanism that significantly mitigates the loss of orthogonality, particularly in ill-conditioned systems. This method sup ports matrices of various dimensions and promotes rapid convergence in the tested scenarios. Comparative studies demonstra te that the proposed technique yields solutions of comparable accuracy while significantly reducing computation time. Numer ical experiments conducted on two structural examples and one random symmetric ill-conditioned matrix confirm that the pro posed technique is both efficient and robust.

Keywords: Eigenproblems; Transformation methods; Householder-QR iteration; Gram-Schmidt.

1. Introduction

The advancement of nonlinear dynamic analysis is evidenced by a substantial body of contemporary research. Predominantly, this research focuses on the temporal integration of nonlinear equations of motion using direct time integration methods, classified as either explicit or implicit [1]. However, these methods are often computationally expensive in real-world applications [2, 3]. In contrast, modal superposition combined with sub-structuring is generally more efficient [3]. Furthermore, modal investigation techniques enhance the physical understanding of nonlinear dynamic responses by enabling analysts to monitor changes in the modal spectrum as deformation progresses. Consequently, modal analysis remains a fundamental approach to dynamic analysis, and numerous studies have sought to improve its efficiency [4].

Nevertheless, conventional modal techniques in nonlinear analysis frequently suffer from computational inefficiencies due to inherent methodological limitations [5]. A significant challenge is posed by ill-conditioned matrices and linear dependencies among the stiffness matrix columns near critical points, which lead to numerical instability and reduced accuracy. Methods based on orthonormalization have demonstrated superior performance in addressing such nonlinearities [6, 7]. For instance, the use of pseudo-forces is a common practice to approximate nonlinear effects; however, this linearization strategy can introduce substantial errors and often requires many iterations for convergence [8].

The QR algorithm, developed by Francis [9, 10] and Kublanovskaya [11], remains a fundamental tool for eigenvalue extraction relying on vector-to-vector operations. Yet, in large-scale applications, weak convergence and stagnation in the presence of clustered eigenvalues necessitate more refined methodologies. The Householder-QR-Inverse iteration method effectively computes eigenvalues and eigenvectors by performing an initial QR decomposition into orthogonal and upper-triangular matrices to ensure stability. Despite these benefits, Householder transformations increase the bandwidth of the unreduced matrix sections, which may hamper efficiency in large systems. The resultant growth of nonzero elements beyond the tridiagonal structure complicates further processing, although iterative refinement techniques can mitigate this expansion while preserving accuracy [12].

Additionally, the Gram–Schmidt orthogonalization method offers a simpler implementation for large-scale modal analysis. However, classical variants may experience a deficiency in orthogonality in nearly rank-deficient matrices, and even modified versions are prone to round-off error accumulation [13]. To address these challenges, this paper introduces a novel method for computing eigenvalues in large, ill-conditioned profile matrices. By extending the traditional QR algorithm with an improved Gram–Schmidt approach, this method is designed to integrate iterative re-orthogonalization and error-correction mechanisms, aiming to enhance stability and computational efficiency, especially when dealing with linearly dependent clustered eigenvalues and convergence stagnation.

The process begins with the selection and normalization of the initial column of the matrix to establish the foundational orthonormal vector. In the Modified Gram–Schmidt (MGS) algorithm, each derived orthonormal vector is immediately used to revise all subsequent columns by eliminating its influence. This instant projection ensures that contributions are subtracted immediately upon orthogonalization, preserving strict orthogonality and mitigating round-off errors. This prompt updating is especially advantageous for nearly linearly dependent vectors, as any delay in removing dependencies can lead to the numerical inaccuracies typically observed in the Classical Gram–Schmidt (CGS) method. Consequently, MGS provides improved numerical stability and robustness, ensuring a more consistent collection of orthonormal vectors in practical applications.

2. Mode Superposition

All methodologies for solving eigenvalue problems inherently possess an iterative characteristic. Specifically, addressing the eigenvalue problem in Eq. (1) is equivalent to determining the roots of the characteristic polynomial p(λ) in Eq. (3). The order of this polynomial is commensurate with the order of the stiffness and mass matrices, denoted by K and M, respectively [12].

$$K\emptyset = \lambda M \emptyset \tag{1}$$
$$p(\lambda) = 0 \tag{2}$$
$$p(\lambda) = det(K - \lambda M) \tag{3}$$

Equation (3) represents the characteristic polynomial of the corresponding constraint problem related to $K\emptyset = \lambda M \emptyset$.

The efficacy of a particular solution methodology is predominantly influenced by two principal factors: the reliability of the procedure and the computational cost. The cost is fundamentally dictated by the volume of high-speed computational operations performed in memory. It is imperative that a methodology be reliably implemented; that is, for well-defined stiffness and mass matrices, the solution should be achieved with the required accuracy without breakdown. In practical scenarios, a failure in the solution process typically occurs only when the problem is inadequately defined, such as due to errors in data input or an inherent lack of specification in the K and M matrices. Algorithms must be evaluated within the framework of these considerations [12].

3. The QR Method with Householder Transformation

A significant approach for transformation solutions is the Householder-QR-Inverse iteration method. This section examines the problem $K\emptyset = \lambda M\emptyset$, where K may possess zero eigenvalues. In this technique, the matrix is first converted into a tridiagonal structure without the need for iteration. This transformed matrix is then utilized within QR iterative methods to compute all eigenvalues.

To ascertain the eigenvalues and eigenvectors of a given matrix K, Householder transformations are initially employed to reduce K to a tridiagonal configuration, simplifying subsequent calculations. QR iteration is then applied to this tridiagonal matrix to extract all eigenvalues. Once these are determined, inverse iteration is used to compute the corresponding eigenvectors of the tridiagonal matrix. Finally, these eigenvectors are transformed back to derive the eigenvectors of the original matrix K, completing the eigensystem calculation [14].

3.1. The Householder Transformation

The fundamental concept is to reduce K matrix to tridiagonal form using reflection matrices $P_k = I - \theta v_k v_k^T$:

$$K_2 = P_1^T K_1 P_1 \tag{4}$$
$$K_3 = P_2^T K_2 P_2 \quad ; \quad K_{n+1} = P_k^T K_k P_k \quad ; \qquad k = 1, \dots , n-2 \tag{5}$$
$$\theta = \frac{2}{v_k^T v_k} \tag{6}$$

A drawback of Householder transformations is the increase in bandwidth within the unreduced segment o K_{n+1}. Consequently, during the reduction process, the original banded structure of matrix K cannot be efficiently exploited.

3.2. The QR Iteration

The QR algorithm is executed on the tridiagonal matrix derived from the Householder transformation of K.

$$K = QR \tag{7}$$

where Q represents an orthogonal matrix and R denotes an upper triangular matrix. Subsequently, we construct:

$$RQ = Q^T KQ \tag{8}$$

In practical applications, it is more efficient to transform the matrix K into an upper triangular configuration using Jacobi rotation matrices; specifically:

$$P_{n,n-1}^T \dots P_{3,1}^T P_{2,1}^T K = R \tag{9}$$
$$Q = P_{2,1}^T P_{3,1}^T \dots P_{n,n-1}^T \tag{10}$$

The QR iteration algorithm is based on the iterative execution of $K = QR$ and $RQ = Q^T KQ$. Using the notation $K_1 = K$, we form:

$$K_k = Q_k R_k \tag{11}$$

And then:

$$K_{k+1} = R_k Q_k \tag{12}$$

The eigenvalues and eigenvectors are then determined as:

$$K_{k+1} \rightarrow \lambda \tag{13}$$

$$Q_1 \dots Q_{k-1}\, Q_k \rightarrow \emptyset \tag{14}$$

The factorization of $K = QR$ can be obtained using the Gram-Schmidt orthogonalization procedure. However, the traditional Gram-Schmidt method suffers from limitations, such as an inefficient re-orthogonalization process. In the following section, we propose a QR method (GRAM-QR) using a Modified Gram-Schmidt process, designed for various scenarios, including ill-conditioned and linearly dependent systems.

4. The Proposed QR Iteration Solution (GRAM-QR)

In certain computational scenarios, the necessity for an explicit QR factorization supersedes the utilization of a sequence of orthogonal matrices, as seen in the Householder variant. The Gram-Schmidt process offers a more explicit form of QR factorization, catering to specific computational requirements in modal analysis [15]. The general decomposition is expressed as:

$$K = QR = Q_1 \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \tag{15}$$

In vector form, the relationship between the initial matrix columns $a_j$and the resulting orthonormal vectors q_j is given by:

$$\left(|a_j| \dots \dots |a_n|\right) = \left(|q_1| \dots \dots |q_{j-1}| \right) \begin{pmatrix} r_{0,0} & \cdots & r_{0,n-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_{n,n-1} \end{pmatrix} \tag{16}$$

In the j^{th}step, the j^{th} column of Q (represented by q_j)) and j^{th} row of R (indicated by r_j) are determined. The orthogonalization of subsequent vectors $a_{j+1} \dots a_n$ is achieved using the orthogonal projector $(I - q_j {q_j}^T)$.

$$\left(I - q_j {q_j}^T \right) a_n = q_j r_{jn} + \cdots + q_n r_{nn} \tag{17}$$

Where the coefficients are computed as:

$$r_{jn} = {q_j}^T a_n \tag{18}$$

In the Modified Gram-Schmidt (MGS) approach, once an orthonormal vector q_j is attained, it is instantly projected out of the remaining column vectors. This sequential projection is represented as:

$$r_{2n} = {q_2}^T (I - q_1 {q_1}^T) a_k \tag{19}$$

4.1. Limitations of Standard MGS in Ill-Conditioned Systems

While the MGS process described in Eqs. (17-19) is more stable than the classical approach, it remains susceptible to orthogonality loss when dealing with ill-conditioned matrices, which are common in structural dynamics with clustered eigenvalues. In such cases, the norm of the residual vector after projection becomes dangerously small, leading to numerical instability during normalization.

4.2. The Proposed GRAM-QR Enhancement: Sensing and Correction

To overcome these limitations, we introduce a Sensing-based Correction mechanism into the MGS framework. The proposed GRAM-QR method modifies the standard loop by introducing a stability threshold τ.

The modified procedure is as follows:

1. Sensing Step: After the projection defined in Eq. (19), the algorithm monitors the norm of the resulting vector v_k.

2. Detection: If $|\, \boldsymbol{v_k} \,| < \tau$,, the algorithm detects a potential loss of linear independence due to the high condition number of the matrix.

3. Local Correction: Instead of proceeding with a potentially unstable normalization, a refined re-orthogonalization step is triggered. The vector is projected again against the previously computed q vectors to eliminate accumulated rounding errors:

$$v_k^{corrected} = v_k - \sum_{i=1}^{j-1} (q_i^T\, v_k)\, q_i \tag{20}$$

This integrated sensing-and-correction loop helps the Q matrix maintain a high level of orthogonality even in the presence of severely ill-conditioned systems, significantly improving the convergence of the eigenvalue problem.

Algorithm 1: GRAM-QR Orthogonalization Algorithm
Input: Matrix $\mathbf{A} \in \mathrm{R}^{\mathrm{m \times n}}$, Stability Threshold τ, Machine precision ϵ.
Output: Orthonormal Matrix **Q**, Upper Triangular Matrix R.

1. Initialize: $\mathbf{Q} = 0_{\mathrm{m \times n}}$, $\mathbf{R} = 0_{\mathrm{n \times n}}$
2. For j = 1 to n:
3. Set $\boldsymbol{v}_j = \boldsymbol{\alpha}_j$ (Initial column vector $\boldsymbol{A}$)
4. For $i = 1$ to $j - 1$:
5. Compute $r_{ij} = \boldsymbol{q}_i^T \boldsymbol{v}_j$
6. Update $\boldsymbol{v}_j = \boldsymbol{v}_j - r_{ij} \boldsymbol{q}_i$ (Modified Gram-Schmidt Projection)
7. End For
8. Sensing and Detection Step
9. If $\|\boldsymbol{v}_j\| < \tau$ then
10. Local Correction Strategy (Re-orthogonalization)
11. For $i = 1$ to $j - 1$:
12. Compute $\delta_{ij} = \boldsymbol{q}_i^T v_j$
13. Update $v_j = v_j - \delta_{ij} \boldsymbol{q}_i$
14. Update $r_{ij} = r_{ij} + \delta_{ij}$ (Update R to maintain $\boldsymbol{A} = \boldsymbol{QR}$)
15. End For
16. End If
17. Compute $r_{jj} = \|\boldsymbol{v}_j\|$
18. If $r_{jj} > \epsilon$ (where ϵ is machine precision) then
19. $\boldsymbol{q}_j = v_j / r_{jj}$
20. Else
21. $\boldsymbol{q}_j = 0$ (Handle linearly dependent case)
22. End If
23. End For
24. Return $\boldsymbol{Q}, \boldsymbol{R}$

5. Numerical Stability and Orthogonality Analysis

The QR algorithm, extensively employed for the evaluation of eigenvalues, may encounter forward instability when applied to ill-conditioned symmetric tridiagonal matrices. This is particularly evident in scenarios where shifts approach eigenvalues characterized by diminutive normalized eigenvector components, potentially leading to premature deflation and unsuccessful convergence [13, 16]. Furthermore, when applied to linearly dependent matrices, the QR iteration technique employing Householder reflections often experiences numerical instability due to failures in orthogonalization.

Specifically, the Householder reflector may approach a near-zero value when the columns of the matrix demonstrate dependence, culminating in significant computational inaccuracies. The presence of rank-deficient matrices exacerbates this predicament, frequently resulting in the premature cessation of the QR iteration process due to the occurrence of near-zero singular values, which adversely impacts the reliable extraction of eigenvalues. The primary challenge is predominantly attributed to rank deficiency, which engenders numerical instability.

5.1. Analysis of Breakdown in Householder Reflection

The Householder vector v is computed as:

$$v = sign(x_1)\|x\|e_1 + x \tag{21}$$

If A possesses dependent columns, the norm of x, $\|x\|$, may become exceptionally small, leading to $v \approx 0$. This results in an unstable normalization process:

$$v' = \frac{v}{\| v \|} \tag{22}$$

For a rank-deficient matrix, where $\| v \| \approx 0$, the division by a near-zero quantity amplifies numerical errors, leading to a loss of precision in the resulting orthogonal matrix.

5.2. Orthogonality Loss in QR Iteration

The QR update follows:

$$A_{k+1} = R_k Q_k \tag{23}$$

In the presence of rank deficiency, the off-diagonal norm may appear small prematurely:

$$Off - Diagonal\ Norm = \| A - diag(diag(A)) \|_F$$

This can falsely indicate convergence, prevent the proper extraction of the remaining eigenvalues and disrupt the iterative process.

5.3. Impact of Rank Deficiency on Singular Values

In a dependent matrix, $A^T A$ has singular values approaching zero:

$$\lambda min(A^T A) \approx 0 \tag{24}$$

This leads to a poorly conditioned matrix, which inherently disrupts the standard QR decomposition and compromises the stability of the eigenvalue computation.

5.4. Stability of the Modified Gram-Schmidt Approach

The proposed GRAM-QR method effectively addresses linear dependence in QR decomposition by integrating the sensing and correction mechanism detailed in Section 4.2. Unlike standard orthogonalization methodologies that may falter in the presence of rank deficiency, the inclusion of the stability threshold τ allows the algorithm to explicitly detect and eliminate components that are collinear with previously calculated vectors, thereby averting singularities and upholding orthogonality throughout the procedure.

By adaptively triggering the local correction strategy only when $\|v_k\| < \tau$, the technique sustains numerical separation without incurring the full computational cost of repeated re-orthogonalization for every vector. This ensures that even in instances where the matrix possesses dependent columns, the transformation remains stable, facilitating valid eigenvalue extraction. Consequently, the GRAM-QR approach demonstrates resilience in addressing computational challenges related to nearly singular matrices and rank-deficient scenarios.

6. Illustrative Example Solutions

This section presents a series of numerical examples designed to evaluate the performance of the proposed GRAM-QR method in terms of CPU time, computational overhead, and the off-diagonal norm per iteration. To ensure a comprehensive validation, two distinct testing scenarios were employed: two structural engineering examples and one randomly generated symmetric ill-conditioned matrix. The primary objective is to elucidate the efficacy and robustness of the enrichment scheme through these illustrative cases.

All simulations were implemented in MATLAB R2021a on a workstation equipped with an Intel(R) Core(TM) i7-6700HQ CPU (2.60 GHz) and 16.0 GB of RAM, operating under Windows 10 (64-bit). The computational efficiency of the proposed GRAM-QR method is benchmarked against the Jacobi and QR-Householder approaches to quantify the improvements in convergence and stability. Furthermore, to establish a baseline for absolute accuracy, the Reference Solution values provided in the comparative tables were computed using the high-precision `Eig` function of MATLAB. As this function is based on the industry-standard LAPACK library and the QZ algorithm, it provides the most reliable reference values for verifying the numerical precision of the results.

6.1. Example-1: Solution of Plane Frame

The first example addressed is the pin-jointed plane framework illustrated in Fig. 1. This configuration comprises a total of thirty-six joints and seventy-two members. Each member located in the region ABEG possesses a specific cross-sectional area of $0.00258\ m^2$, whereas those within section CDFHKJGE exhibit a cross-sectional area of $0.00129\ m^2$. The elastic modulus applicable to all members is uniformly maintained $2 * 10^5$MN/m^2 and joints numbered 1 through 5 are entirely pinned. The structural framework is subjected to a vertical load of $0.1\ MN$ acting at joint 36, (This problem is taken from Ref. [17]).

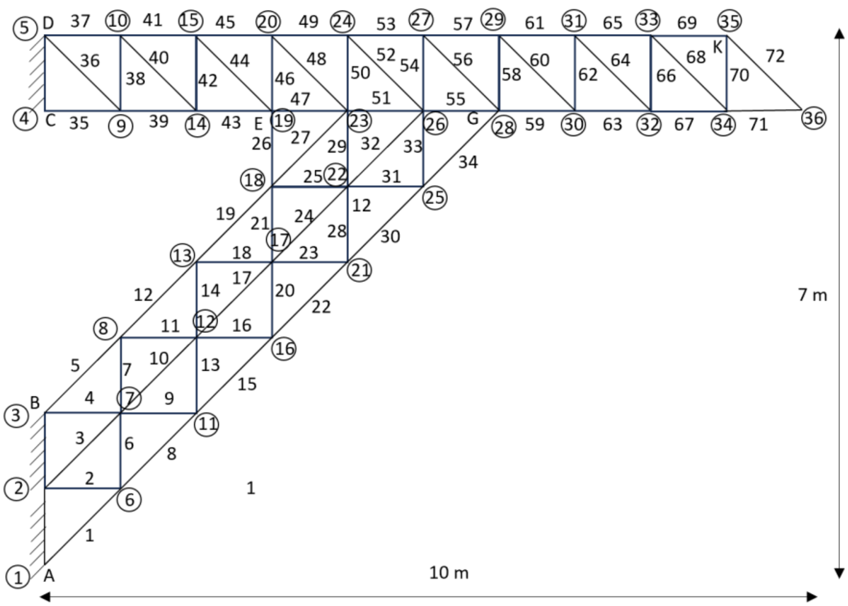

Fig. 1. Pin-Jointed plane frame example

The eigenvalue estimates obtained via the Jacobi, QR-Householder, and the proposed GRAM-QR methods exhibit excellent agreement with the Reference solution (Table 1). Nonetheless, marked disparities are observed in terms of computational efficiency. Specifically, the Jacobi method necessitated 63 iterations and 5.8652 seconds of CPU time, while the QR-Householder approach achieved convergence within 32 iterations and 3.4236 seconds. Notably, the proposed GRAM-QR method, employing an effective instant projection and saving strategy, required only 13 iterations and 1.2985 seconds to converge. These findings underscore that, although all three methods yield comparable eigenvalue accuracy, the instant projection and saving strategy significantly enhances performance by reducing both the number of iterations and the total computational time.

Table 1. Eigen Values for Example 1: Plane Frame

EIGEN VALUES $(*10^3)$	JACOBI	QR-HOUSEHOLDER	GRAM-QR	Reference Solution
λ_1	2.6627	2.6627	2.6627	2.6627
λ_2	2.3983	2.3971	2.4111	2.4111
λ_3	2.3614	2.3625	2.3586	2.3486
λ_4	2.1169	2.0779	2.1218	2.1223
λ_5	2.0651	2.1085	2.0964	2.0640
λ_6	1.9502	1.946	1.9459	1.9459
λ_7	1.7236	1.7237	1.7243	1.7258
λ_8	1.7155	1.7157	1.7149	1.7146
λ_9	1.6488	1.6478	1.6477	1.6483
λ_{10}	1.5796	1.5764	1.5793	1.5793
Total Iterations	63	32	13	
CPU TIME (sec)	5.8652	3.4236	1.2985	

6.2. Example-2: Solution of Strut Structure

Thompson and Hunt [18] conducted an examination of this particular structural configuration in Fig.2 with respect to its sensitivity to imperfections regarding the buckling behavior of a reticulated or triangulated column. This structural assembly consists of thirty-five members, each featuring a circular cross-section and possessing a uniform modulus of elasticity quantified at $E = 68.964 \times 10^{6} kN/m2$. The cross section-area of members 1–21 is 54.84 cm^2 and members 22–35 is 51.61 cm^2.

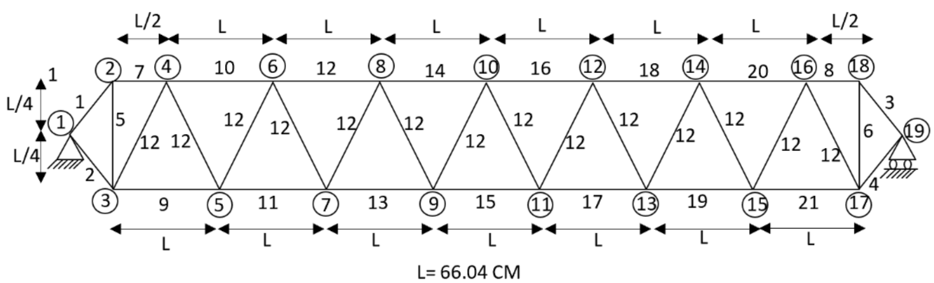

Fig. 1. Thompson's strut truss

The results presented in Table 2 are consistent with those observed in the previous numerical example, further reinforcing the robustness of the underlying trend. This uniformity across different examples strongly suggests that the methodology and the phenomena under investigation are highly reproducible. The replication of similar eigenvalue patterns, iteration counts, and CPU times substantiates the generalizability of our approach and confirms that the observed behavior persists under varied conditions.

Table 2. Eigen Values for Example 2: Thompson's strut truss

EIGEN VALUES $(*10^4)$	JACOBI	QR-HOUSEHOLDER	GRAM-QR	Reference Solution
λ_1	3.3054	3.3146	3.339	3.339
λ_2	3.2511	3.2649	3.3099	3.3099
λ_3	3.1285	3.1542	3.2532	3.2725
λ_4	3.1054	3.1346	3.2011	3.2109
λ_5	3.3059	3.3065	3.0857	3.0887
λ_6	2.8742	2.8974	3.0356	3.0356
λ_7	3.0545	2.9562	2.7732	2.7844
λ_8	2.7684	2.7756	2.7537	2.7637
λ_9	2.4314	2.4294	2.4380	2.435
λ_{10}	2.3365	2.4126	2.4102	2.4062
Total Iterations	71	41	19	
CPU TIME (sec)	8.1258	5.7202	2.8934	

6.3. Example-3: Random-Symmetric Ill-Conditioned Matrix

In this example, we investigate the performance of the proposed method using a randomly generated symmetric ill-conditioned matrix of size 400×400. To ensure reproducibility and precise control over the numerical stability, the matrix was constructed using a Singular Value Decomposition (SVD)-based approach. Specifically, the matrix was synthesized by selecting a set of singular values that precisely define a condition number of 2.8920×10^{15}, thereby creating a system that is nearly linearly dependent.

Such severe ill-conditioning is intimately related to the phenomenon of clustered eigenvalues, where multiple eigenvalues are extremely close in value, a situation frequently observed in physical systems, such as vibrations or oscillations with nearly identical natural frequencies. The clustering of eigenvalues poses significant challenges for traditional eigenvalue computation techniques, such as the standard QR method, often leading to stagnation in convergence and difficulties in accurately resolving both eigenvalues and their respective eigenvectors. By evaluating Example 3 under these adverse conditions, we are able to rigorously assess the robustness of the proposed approach, demonstrating its capability to maintain accuracy and efficiency even when faced with the numerical complexities induced by severe clustering.

Table 3. Off-diagonal norm per iteration for Example 3 (Random-Symmetric Ill-Conditioned Matrix with Linearly Dependent Columns) Condition number: $\mathbf{2.8920 \times 10^{15}}$

Number of iterations	JACOBI	QR-HOUSEHOLDER	GRAM-QR
1	5.0123	4.9105	4.6105
3	4.7458	2.1457	1.9457
6	4.6905	0.6952	0.6352
9	4.669	0.3987	0.2402
12	4.6523	0.3765	0.0936
15	4.6421	0.3665	0.0358
18	4.6376	0.366	0.0132
21	4.6379	0.3759	0.0045
24	4.6426	0.396	0.0014
27	4.651	0.4272	0.0004
30	4.6625	0.4695	0.00005
CPU TIME(sec)	Not converged	Not converged	5.8265

The numerical results and the corresponding convergence behavior are presented in Table 3 and Fig. 3. Table 3 compares the off-diagonal norm reduction across iterations for the three methods, while Fig. 3 visually illustrates the rate of convergence. The Jacobi method exhibits minimal progress, with the off-diagonal norm decreasing only slightly from 5.0123 at iteration 1 to 4.6625 by iteration 30, indicating extremely slow convergence. The QR–Householder method shows a more rapid initial reduction but ultimately suffers from numerical stagnation, failing to converge within the tested iterations. In contrast, the proposed GRAM-QR method achieves a substantial reduction, from 4.6105 at iteration 1 to a near-zero value of 0.00005 by iteration 30, converging within 5.8265 seconds.

This superior performance, as clearly depicted in both the data and the convergence curve, is attributed to the sensing-and-correction mechanism, which prevents the loss of orthogonality that typically leads to stagnation in standard QR transformations when dealing with highly ill-conditioned matrices. By adaptively triggering the re-orthogonalization process, the GRAM-QR method effectively mitigates the impact of clustered eigenvalues and round-off errors. This clearly demonstrates the robustness and efficiency of the proposed approach in handling the numerical challenges associated with nearly singular matrices.

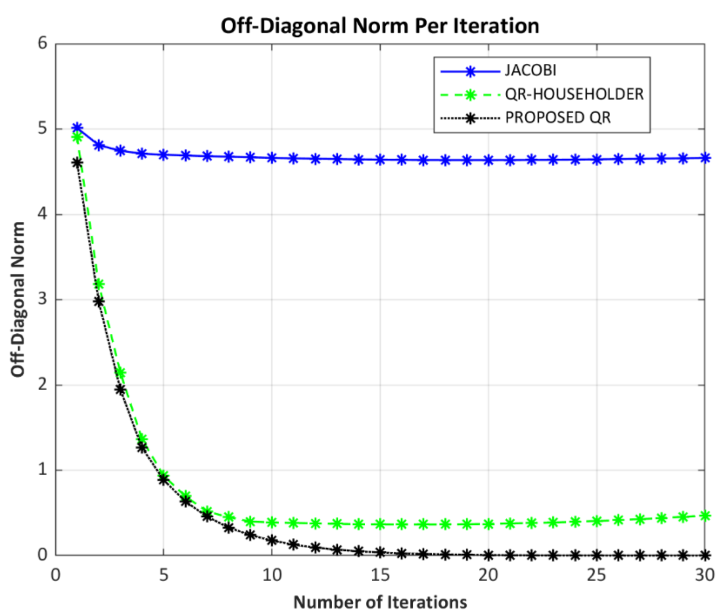

Fig. 3. The off-diagonal norm at each iteration, used to compute the eigenvalues, demonstrates the convergence rate of algorithms

7. Conclusion

A modified iteration of the Gram-Schmidt process has been employed as the foundational element in the QR algorithm, facilitating the generation of orthogonal matrices characterized by enhanced stability and immediate projection capabilities. By incorporating a sensing-and-correction mechanism, the proposed methodology demonstrates an improvement in accuracy and a significant augmentation in computational efficiency within the analyzed cases. Through a series of numerical investigations involving structural case studies and a singular random linearly dependent matrix, the proposed approach has demonstrated a substantial reduction in computational costs while achieving high precision in alignment with reference eigenvalue solutions.

The numerical results reveal that the superior efficiency of the GRAM-QR method is primarily driven by a marked reduction in the total number of iterations required for convergence, effectively mitigating the computational overhead typical of standard transformation methods. Furthermore, the outcomes for ill-conditioned stiffness matrices, common in nonlinear dynamic analyses, demonstrate a critical advancement in modal methods, particularly in ensuring numerical stability and overcoming the stagnation and non-convergence issues that often render conventional techniques impractical in real-world applications. Consequently, this research proposes a robust and efficient framework that can be effectively utilized for addressing complex eigenvalue problems, particularly those manifesting in the context of geometric nonlinearities.

Author Contributions

Both authors were equally involved in all stages of the research, from the initial design and implementation to the writing and editing of the final manuscript.

Conflict of Interest

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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